A CLASS OF COMPLETELY SOLVABLE GAMES

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A pair (P, φ) of a set P and a map $\varphi\colon P \longrightarrow 2^P$ from P to the set 2^P of subsets of P is called a *game*. If $p, q \in P$ and $q \in \varphi(p)$ we write $p \to q$. Thus, a game, in our sense, is nothing but a system of arrows between elements of a set. A game (P, φ) is said to be *finite*, if there exists no infinite sequence $p_0, p_1, \ldots, p_n, \ldots$ of elements of P satisfying $p_i \to p_{i+1}$ for any $i \geq 0$. A finite game (P, φ) may be considered as a model of a 2-player (impartial [1]) game as follows:
An element $p_0 \in P$ is fixed as an opening position. The first player chooses a position $p_1 \in \varphi(p_0)$, the second player chooses next position $p_2 \in \varphi(p_1)$, and the first chooses a $p_3 \in \varphi(p_2)$, ... Since (P, φ) is a finite game, the players inevitably encounter a non-negative integer l such that $\varphi(p_l) = \emptyset$. If l is odd (resp. even), we say that the first (resp. second) player *wins*. (In other words, a player who is unable to continue its play *looses*.)
For a fixed p_0, one of the players has a winning strategy by Zermelo's theorem. But, in general, it is not easy to answer the following practical questions :
(1) Which player has a winning strategy? (2) How can we find out good moves?
This is particularly so in a 'transfinite' case, in which the players may have an infinite number of choices on their turn. Hence it might be of interest to construct a class of *completely solvable games*, i.e. a class of finite games such that one can carry out a calculation of small size to know, for a given opening position of the game, which player has a winning strategy and which moves are good. The main result of this talk is :

Finite 'plain' games are completely solvable.

Plain games are defined axiomatically; Nim and Sato-Welter game (=Welter's game [1]), and their transfinite analogues, are examples of finite plain games. The main result remains true even if the above mentioned definitions of winner and looser are interchanged, namely under the misère rule in the terminology of [1]. A close relationship with the work of Proctor [3] will also be shown. For some (older) results related to this talk, see [2].

References

[1] J. H. Conway, *On Numbers and Games*, Second Edition, A K Peters, 2001.

[2] N. Kawanaka, *Games and algorithms with hook structure*, Sugaku Expositions **28**, 73–93 (2015). (Japanese version :「数学」**63**-4, 421–441 (2011).)
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[3] R. A. Proctor, *Dynkin diagram classification of λ-miniscule Bruhat lattices and of d-complete posets*, J.Algebraic Combin., **9**, 61–94 (1999).

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